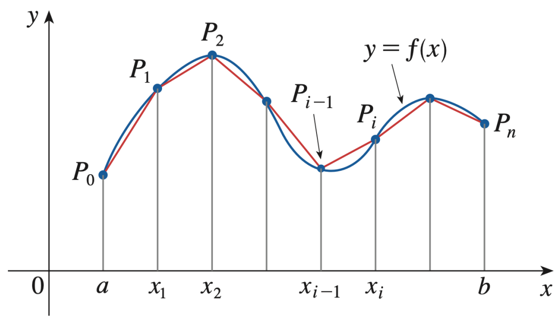
8.1 / 10.2 Arc Length

Theorem (Arc Length Formula)**.** Let $y = f(x)$ be a curve defined on the interval $[a, b]$, and suppose that $f'(x)$ is continuous on $[a, b]$. The arc length L of the curve is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

1. **Polygonal Approximation:**

- Divide the interval $[a, b]$ into n subintervals of equal width Δx.
- For each i, let $y_i = f(x_i)$ and consider the points $P_i = (x_i, y_i)$ on the curve.
- Approximate the curve by a polygonal path connecting these points.



2. **Length of a Single Segment:**

- The length of a segment connecting two consecutive points P_{i-1} and P_i is:

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

- By the Mean Value Theorem, $\Delta y_i = f'(x_i^*)\Delta x$ for some $x_i^* \in [x_{i-1}, x_i]$.
- Substitute this into the segment length:

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

3. **Total Length of the Polygonal Path:**

- Sum the lengths of all segments:

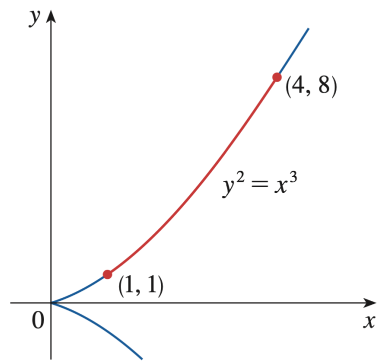
$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \Delta x \sqrt{1 + (f'(x_i^*))^2}$$

4. **Take the Limit as $n \to \infty$:**

- As $n \to \infty$, $\Delta x \to 0$, and the sum becomes a definite integral:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



The top half of the curve is given by $y = x^{3/2}$

Compute $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Let $u = 1 + \frac{9}{4}x$. Then $du = \frac{9}{4}dx$. So $dx = \frac{4}{9}du$.

When $x = 1$, $u = \frac{13}{4}$. When $x = 4$, $u = 10$.

$$= \int_{13/4}^{10} \sqrt{u} \cdot \frac{4}{9}\, du = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10}$$

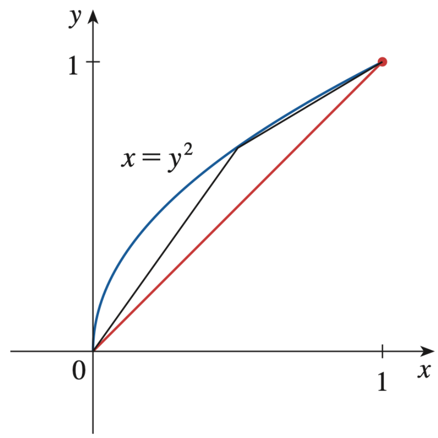
$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$\approx 7.633$$

Theorem. If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example. Find the length of the arc of the parabola $x = y^2$ from $(0,0)$ to $(1,1)$.



$$\frac{dx}{dy} = 2y$$

$$L = \int_0^1 \sqrt{1 + (2y)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

Use the trig substitution $y = \frac{1}{2}\tan\theta$

Then $dy = \frac{1}{2}\sec^2\theta\, d\theta$.

$$\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

When $y = 0$, $\theta = 0$. When $y = 1$, $\theta = \tan^{-1}(2)$

$$L = \int_0^{\tan^{-1}(2)} \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}(2)}$$

$$\approx 1.478$$

Example. Set up an integral for the length of the arc of the hyperbola $xy = 1$ from $(1, 1)$ to $(2, \frac{1}{2})$.

The equation of the curve is $y = \frac{1}{x}$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

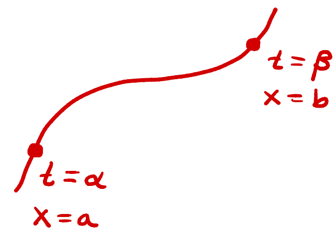
$$L = \int_1^2 \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

Theorem. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Start with the arc length formula for the curve $y = f(x)$

where $a \leq x \leq b$ and $a = f(\alpha)$ and $b = f(\beta)$



$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Changing variables, when $x = a$, $t = \alpha$. when $x = b$, $t = \beta$.

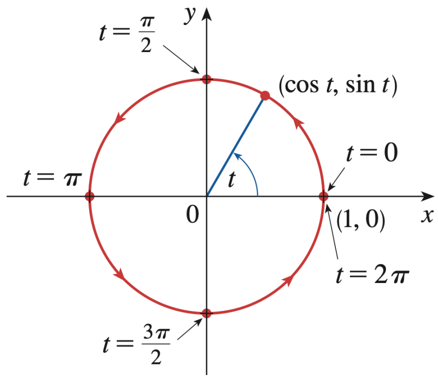
And $dx = \frac{dx}{dt} \cdot dt$

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \cdot \frac{dx}{dt}\, dt = \int_\alpha^\beta \sqrt{\frac{(dx/dt)^2}{(dx/dt)^2} + \frac{(dy/dt)^2}{(dx/dt)^2}} \cdot \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{dx/dt} \cdot \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example. Find the length of the unit circle described by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.



$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = \cos t$$

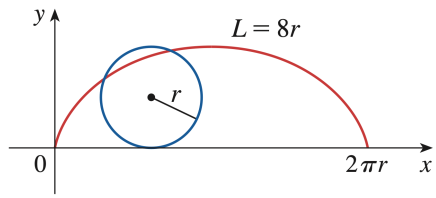
$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= [t]_0^{2\pi}$$

$$= 2\pi$$

Example. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



- One arch of the cycloid corresponds to the parameter interval: $0 \leq \theta \leq 2\pi$.

- Compute the derivatives:

$$x = r\theta - r\sin\theta \qquad y = r - r\cos\theta$$

$$\frac{dx}{d\theta} = r - r\cos\theta = r(1-\cos\theta) \qquad \frac{dy}{d\theta} = r\sin\theta$$

- Substitute into the arc length formula:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_0^{2\pi} r\sqrt{2(1-\cos\theta)}\, d\theta$$

$$1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta$$

- Use the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$:

$$L = \int_0^{2\pi} r\sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 2r\left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

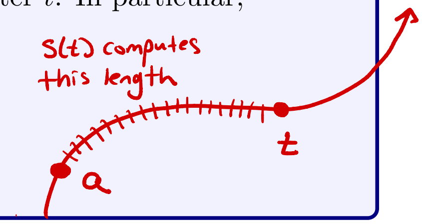
- Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$, and so $\sin(\theta/2) \geq 0$.

$$L = 2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 2r\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$

$$= 2r\left[-2\cos\pi - (-2\cos 0)\right] = 2r\left[2+2\right] = 8r$$

Definition. Suppose a curve is described by $x = f(u)$, $y = g(u)$, where $f'(u)$ and $g'(u)$ are continuous. The arc length function $s(t)$ gives the length of a curve from an initial point $(f(a), g(a))$ to the point $(f(t), g(t))$ corresponding to the parameter t. In particular,

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$



Remark. If parametric equations describe the position of a moving particle, then the speed $v(t)$ is the derivative of the arc length function $s(t)$. Indeed, the speed $v(t)$ is the rate of change of the total distance traveled along a curve with respect to time. By the Fundamental Theorem of Calculus,

$$v(t) = s'(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example. A particle's position is given by $x = 2t + 3$, $y = 4t^2$, $t \geq 0$. Find the speed of the particle when it is at the point $(5, 4)$.

The speed of the particle is

$$v(t) = \sqrt{(2)^2 + (8t)^2}$$

The particle is at $(5,4)$ when $t = 1$

$$v(1) = \sqrt{4 + (8 \cdot 1)^2} = \sqrt{68} \approx 8.25 \text{ units/time}$$